Mahatma Fule Arts, Comm. & Sitaramji Chaudhari Science Mahavidyalaya, Warud.

## Department of Mathematics

# CLASSICAL MECHANICS

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•Mechanics : Effect of forces on material bodies.

Mechanics

Statics

Effect of forces on bodies at rest

Dynamics

Effect of forces on bodies in motion

•Matter : Universe is either matter or energy.
Whatever we feel, see, or perceive around us is a matter. It occupies Space.
Ex. Table, Chair, Air, Water etc.

- **Body** : Portion of matter.
- **Particle** : Infinitesimal portion of matter having position.
- **Rigid body** :A system of particles.
- **Mass** : A quantity of matter.
- **Weight** : w = mg
- **Force** : An external influence which changes state of rest or uniform motion of a body.
- **Resultant force and its components**

$$\bar{R} = \overline{p_1} + \overline{p_2} + \overline{p_3} \ldots \ldots \ldots + \overline{p_n}$$

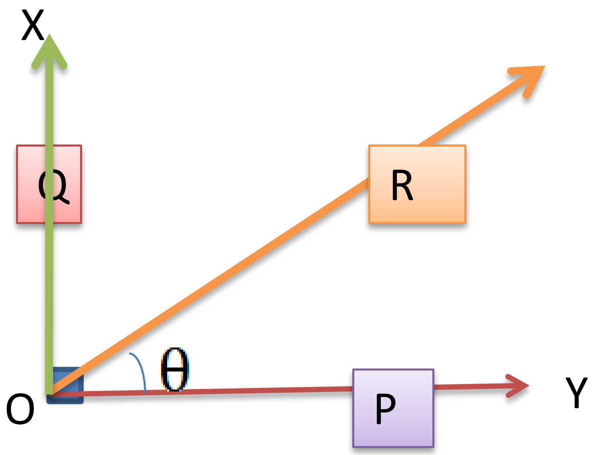
- **Equilibrium of forces**

$$\bar{R} = \overline{p_1} + \overline{p_2} + \overline{p_3} \ldots \ldots \ldots + \overline{p_n} = 0$$

# Resolved Parts of a Force

Let $\overline{R}$ be the resultant of two forces $\overline{P}$ & $\overline{Q}$ acting along OX & OY.

$$\therefore \quad \overline{R} = \overline{P} + \overline{Q} \quad \ldots\ldots\ldots\ldots(1)$$




$$\overline{R} \times \overline{P} = \overline{P} \times \overline{P} + \overline{Q} \times \overline{P}$$

$$|\overline{R} \times \overline{P}| = |\overline{Q} \times \overline{P}|$$

$$\therefore \; R\,P\sin\theta = Q\,P\sin\frac{\pi}{2} \;\Rightarrow\; Q = R\sin\theta$$

Similarly

$$\overline{R} \times \overline{Q} = \overline{P} \times \overline{Q} + \overline{Q} \times \overline{Q}$$

$$|\overline{R} \times \overline{Q}| = |\overline{P} \times \overline{Q}|$$

$$\therefore \; R\,Q\sin(^{\pi}/_{2} - \theta) = P\,Q\sin\frac{\pi}{2} \;\Rightarrow\; P = R\cos\theta$$

Hence, $P = R\cos\theta$ & $Q = R\sin\theta$ are resolved parts of R along OX & OY.

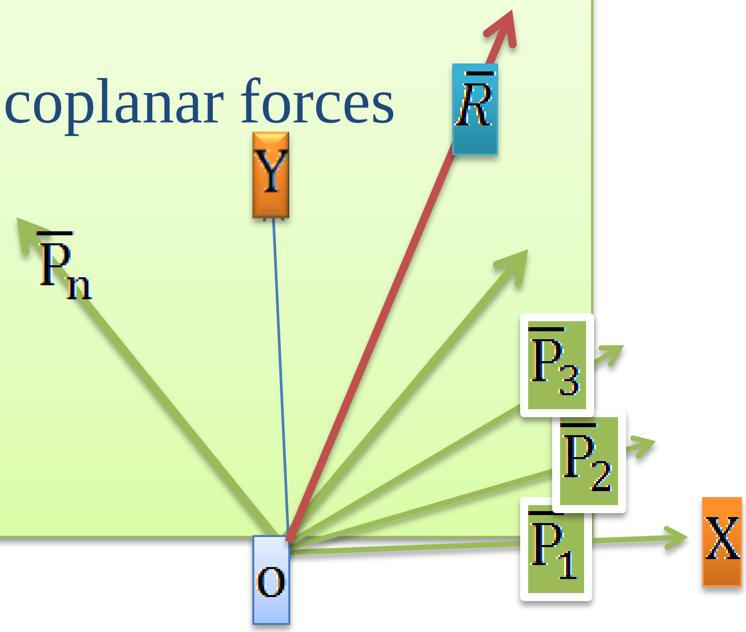
## Resultant of n-coplanar forces

Let $\overline{P_1}, \overline{P_2}, \overline{P_3}, \ldots \ldots \ldots, \overline{P_n}$ be the n – coplanar forces acting upon the particle at o.

Let R be their resultant

Making an angle θ with ox.

Let $\theta_1, \theta_2, \theta_3, \ldots\ldots\ldots \theta_n,$ be the angles made by $\overline{P_1}, \overline{P_2}, \overline{P_3}, \ldots \ldots \ldots, \overline{P_n}$ .




$$\overline{P_1} = P_1 cos\theta_1 i + P_1 sin\theta_1 j$$

$$\overline{P_2} = P_2 cos\theta_2 i + P_2 sin\theta_2 j$$

⋮

$$\overline{P_n} = P_n cos\theta_n i + P_n sin\theta_n j$$

$$\bar{R} = \bar{P}_1 + \bar{P}_2 + \bar{P}_3 + . . . . . + \bar{P}_n$$

$$\therefore \quad \bar{R} = (P_1 \cos\theta_1 + P_2 \cos\theta_2 + \cdots \ldots + P_n \cos\theta_n)i$$

$$+ (P_1 \sin\theta_1 + P_2 \sin\theta_2 + \cdots \ldots + P_n \sin\theta_n)j$$

Since $\bar{R}$ makes an angle $\theta$ with ox

$$\bar{R} = R\cos\theta\, i + R\sin\theta\, j$$

$$\therefore \quad R\cos\theta = \sum_{r=1}^{n} p_r \cos\theta_r = X$$ and $$R\sin\theta = \sum_{r=1}^{n} p_r \sin\theta_r = Y$$

$$\therefore \quad R^2 = \sqrt{X^2 + Y^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{Y}{X}$$

Where,

$$X = P_1 \cos\theta_1 + P_2 \cos\theta_2 + \cdots \ldots . + P_n \cos\theta_n$$

$$Y = P_1 \sin\theta_1 + P_2 \sin\theta_2 + \cdots \ldots . + P_n \sin\theta_n$$

# Lami's Theorem

A particle is in equilibrium under the action of three forces $\overline{P}$, $\overline{Q}$, $\overline{R}$ if and only if,

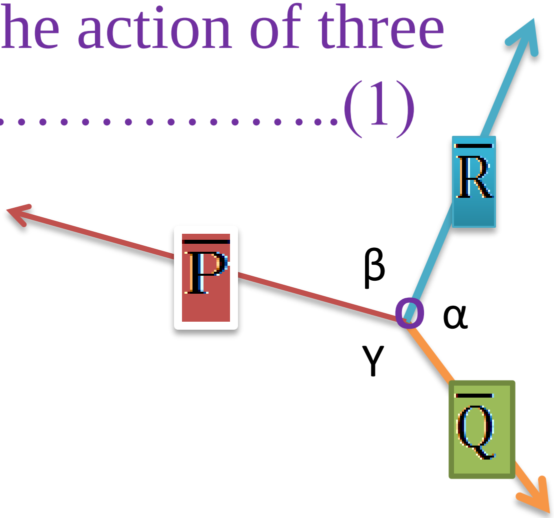
$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

Where,

$$\alpha = \measuredangle(\overline{Q}, \overline{R}), \beta = \measuredangle(\overline{P}, \overline{R}), \text{and } \gamma = \measuredangle(\overline{P}, \overline{Q})$$

Sol. : A particle is in equilibrium under the action of three forces $\overline{P}$, $\overline{Q}$, $\overline{R}$.

$\therefore$ $\overline{P} + \overline{Q} + \overline{R} = 0$ ..................(1)




Multiplying by $\times\overline{P}$

$$\overline{P} \times \overline{P} + \overline{Q} \times \overline{P} + \overline{R} \times \overline{P} = 0$$

$$\overline{Q}\times\overline{P} = -\overline{R}\times\overline{P} \Rightarrow |\overline{Q}\times\overline{P}| = |\overline{R}\times\overline{P}|$$

$\therefore$ $QP\sin\gamma = PR\sin\beta \Rightarrow \dfrac{Q}{\sin\beta} = \dfrac{R}{\sin\gamma}$ ....................(2)

Multiplying (1) by $\times\,\overline{Q}$

$$\overline{P}\times\overline{Q} + \overline{Q}\times\overline{Q} + \overline{R}\times\overline{Q} = 0$$

$$\overline{P}\times\overline{Q} = -\overline{R}\times\overline{Q} \Rightarrow |\overline{P}\times\overline{Q}| = |\overline{Q}\times\overline{R}|$$

$$P\,Q\sin\gamma = Q\,R\sin\alpha \Rightarrow \frac{P}{\sin\alpha} = \frac{R}{\sin\gamma} \quad \text{.............(3)}$$

From (2) and (3) $\dfrac{P}{\sin\alpha} = \dfrac{Q}{\sin\beta} = \dfrac{R}{\sin\gamma}$

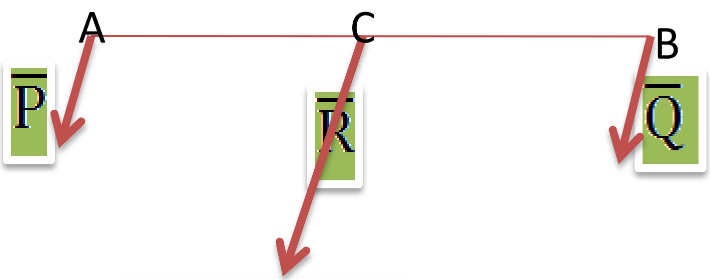
Where, $\alpha = \measuredangle(\overline{Q}, \overline{R})$,

$\beta = \measuredangle(\overline{P}, \overline{R})$

and $\gamma = \measuredangle(\overline{P}, \overline{Q})$

# Parallel Forces

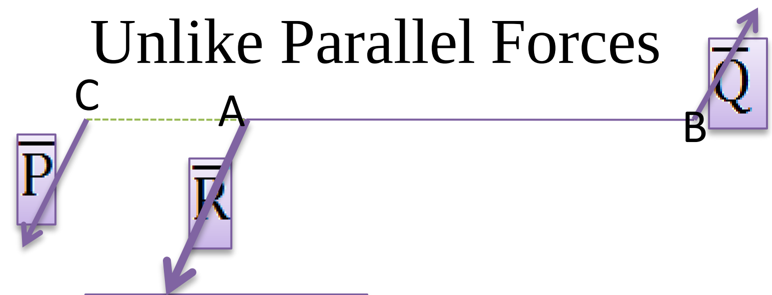
## Like Parallel forces




$$R = P + Q$$

R acts at a point C which divides AB internally such that P AC = Q BC

## Unlike Parallel Forces



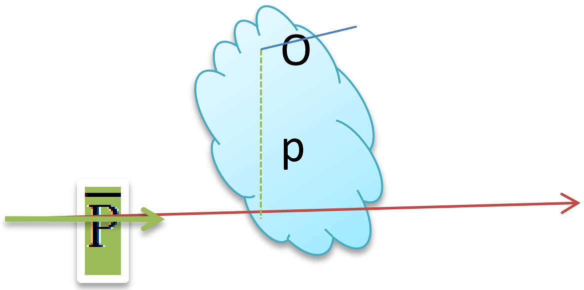

$$R = P - Q$$

R acts at a point C which divides AB externally such that P AC = Q BC

# Moment of force

Moment of force $\overline{P}$ about a fixed point 'o' is defined as

$$G = P\,p$$




**G is positive if rotation is anticlockwise**
**G is negative if rotation is clockwise**

**The moment of force about a fixed point is zero if the line of action of the force passes through it.**
**i.e. $G = P\,p = 0$ iff $p = 0$**

# Vector moment of force

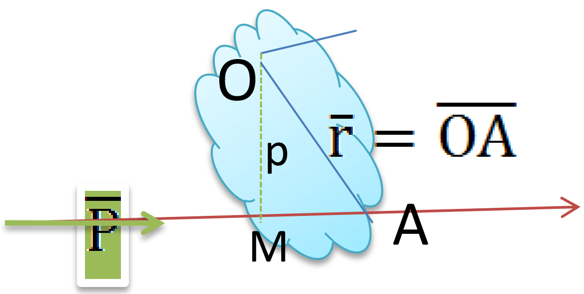
Let $\overline{P}$ be a force acting at point A of a body whose P. V. w. r. to the fixed point be $\bar{r} = \overline{OA}$ .
Then,

$$\bar{r} \times \overline{P} = r\,P \sin\alpha\, \bar{e}$$

$$|\bar{r} \times \overline{P}| = r\,P \sin\alpha$$

$$= P\,p \qquad \text{where } p = r \sin\alpha$$

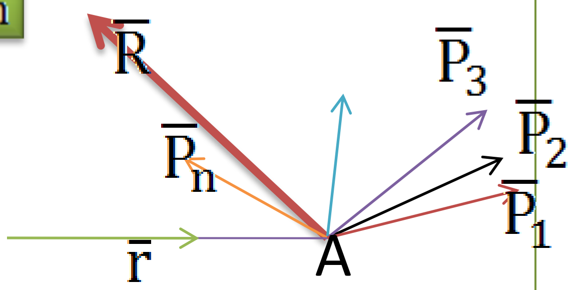
$\therefore$ $|\bar{r} \times \overline{P}| = G \Rightarrow \overline{G} = \bar{r} \times \overline{P}$



Rule -1) : The algebraic sum of the moments of any number of coplanar forces about a point in their plane is equal to the moment of their resultant about that point.

Proof: Let $\overline{R}$ is the resultant of any no. of coplanar forces $\overline{P}_{1}, \overline{P}_{2}, \overline{P}_{3}, \ldots \ldots \overline{P}_{n},$ acting at point A.

i.e. $\overline{R} = \overline{P}_1 + \overline{P}_2 + \overline{P}_3 + \cdots \ldots + \overline{P}_n$




Sum of the moments of forces about O $= \overline{r}x\overline{P}_1 + \overline{r}x\overline{P}_2 + \cdots .. + \overline{r}x\overline{P}_n$

$= \overline{r}\, x\, (\overline{P}_1 + \overline{P}_2 + \overline{P}_3 + \cdots \ldots + \overline{P}_n)$

$= \overline{r}\, x\, \overline{R}$

= Moment of the resultant about O.

# Couple

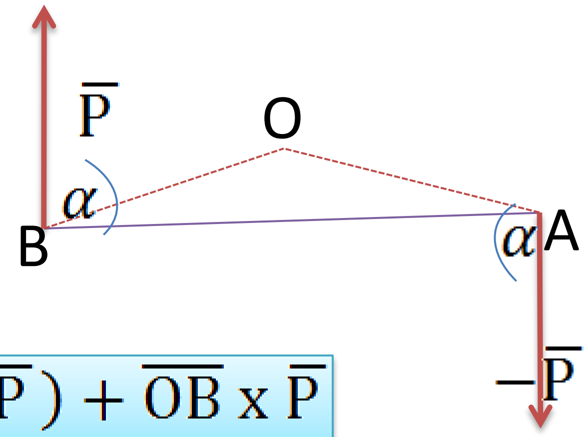
A system of two equal and unlike parallel forces

- **Two equal unlike parallel forces** direction of lines of actions are not same form a **couple.**
- The perpendicular distance between the lines of action of forces forming a couple is called as the **arm of a couple.**

**Let O be a fixed point.**

Moment of $-\bar{P}$ about O $= \overline{OA} \times (-\bar{P})$

Moment of $\bar{P}$ about O $= \overline{OB} \times \bar{P}$




Sum of the moment of the forces forming a couple

$$= \overline{OA} \times (-\bar{P}) + \overline{OB} \times \bar{P}$$

$$= (\overline{OB} - \overline{OA}) \times \bar{P}$$

$$= \overline{AB} \times \bar{P}$$

The sum of the moment of the forces about a fixed point ‘o’ is defined as the moment of couple

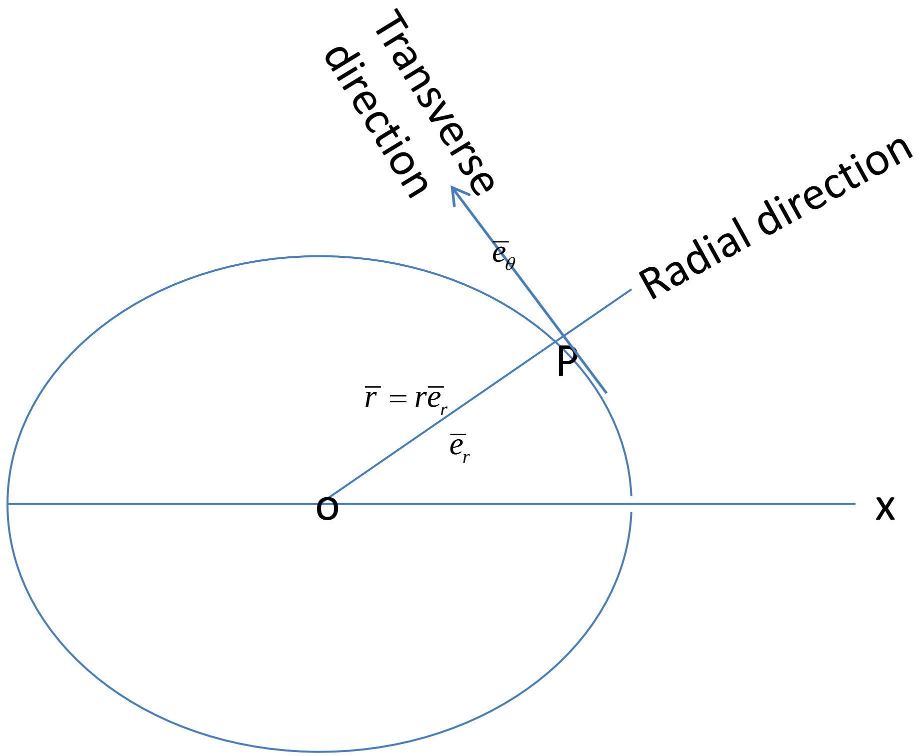
$$|\overline{G}| = |\overline{AB} \times \overline{P}|$$

$= AB\ P \sin\alpha$

$= P\ p$

$G = P\ AB$ where $p = AB$ the perp. distance between the lines of action of the forces. i.e. the Arm of the couple.

# Radial and Transverse Accelerations




$$O\overline{P} = \overline{r} = xi + yj$$

$$\Rightarrow velocity, \overline{V} = \dot{\overline{r}} = \frac{d\overline{r}}{dt} = \dot{x}i + \dot{y}j$$

$$and\ \ acceleration\ \ \overline{a} = \frac{d\overline{v}}{dt} = \ddot{\overline{r}} = \ddot{x}i + \ddot{y}j$$

Theorem: Prove that $\frac{d}{dt}\overline{e}_r = \overline{e}_\theta \frac{d\theta}{dt}$ and $\frac{d}{dt}\overline{e}_\theta = -\overline{e}_r \frac{d\theta}{dt}$

Proof: Let

$$\overline{e}_r = \cos\theta i + \sin\theta j \quad and \quad \overline{e}_\theta = -\sin\theta i + \cos\theta j$$

$$\therefore \frac{d}{dt}\overline{e}_r = -\sin\theta i(\dot{\theta}) + \cos\theta i(\dot{\theta}) = (-\sin\theta i + \cos\theta j)(\dot{\theta}) = \overline{e}_\theta \frac{d\theta}{dt}$$

$$And \quad \frac{d}{dt}\overline{e}_\theta = -\cos\theta i(\dot{\theta}) - \sin\theta i(\dot{\theta}) = -(\cos\theta i + \sin\theta j)(\dot{\theta}) = -\overline{e}_r \frac{d\theta}{dt}$$

$$\therefore \frac{d}{dt}\overline{e}_r = \overline{e}_\theta \frac{d\theta}{dt} \quad and \quad \frac{d}{dt}\overline{e}_\theta = -\overline{e}_r \frac{d\theta}{dt}$$

**Obtain the expressions for Radial and transverse accelerations**

$$Let\ \overline{OP} = \overline{r} = r\ \overline{e}_r \ ....................................(1)$$

$$where\ \overline{e}_r\ is\ the\ unit\ vector\ along\ radial\ direction.$$

$$\Rightarrow \overline{V} = \dot{\overline{r}} = r\frac{d}{dt}\overline{e}_r + \overline{e}_r\,\dot{r} = r[\overline{e}_\theta\,\dot{\theta}] + \overline{e}_r\,\dot{r} \qquad , \quad where\ \frac{d}{dt}\overline{e}_r = \overline{e}_\theta\frac{d\theta}{dt}$$

$$\overline{V} = \dot{\overline{r}} = \dot{r}\overline{e}_r + r\dot{\theta}\overline{e}_\theta \ ...........................(2) \qquad \therefore Speed \quad V = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$$

$$\overline{a} = \frac{d\overline{V}}{dt} = \ddot{\overline{r}} = \dot{r}\frac{d}{dt}\overline{e}_r + \overline{e}_r\,\ddot{r} + r\dot{\theta}\frac{d}{dt}\overline{e}_\theta + r\overline{e}_\theta\ddot{\theta} + \dot{\theta}\overline{e}_\theta\,\dot{r}$$

$$= \dot{r}\,\overline{e}_\theta\,\dot{\theta} + \overline{e}_r\,\ddot{r} + r\dot{\theta}(-\overline{e}_r\dot{\theta}) + r\ddot{\theta}\overline{e}_\theta + \dot{r}\dot{\theta}\overline{e}_\theta$$

$$\overline{a} = (\ddot{r} - r\dot{\theta}^2)\overline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\overline{e}_\theta$$

$$\overline{a} = (\ddot{r} - r\dot{\theta}^2)\overline{e_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\overline{e_\theta}$$

$$Now \ \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = \frac{1}{r}\left[r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}\right] = \left[r\ddot{\theta} + 2\dot{r}\dot{\theta}\right]$$

$$\therefore \overline{a} = (\ddot{r} - r\dot{\theta}^2)\overline{e_r} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\overline{e_\theta}$$

*Hence,*

$$Radial\ acceleration\ a_r = \ddot{r} - r\dot{\theta}^2$$

$$And\ Transverse\ acc\ a_\theta = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$$

$$Total\ acc\ a = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + \left[\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\right]^2}$$

*If $\phi$ is the angle made by trans. acc. with radial acc. then,*

$$\tan\phi = \frac{\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)}{\ddot{r} - r\dot{\theta}^2} \qquad i.e. \tan\phi = \frac{a_\theta}{a_r}$$

# Lagrangian Dynamics

## Degrees of Freedom:

## Generalized Coordinates and Velocities

## Constraints

**Force of constraints does no work in any possible displacement**
**Let $\bar{R}$ be the reaction of the surface (or force of constraints)**
**Then $\bar{R}$ is along the normal to the surface. Hence orthogonal to the displacement vector $d\bar{r}$ .**

$$\therefore \ \bar{R} \bullet d\bar{r} = 0$$

**i.e. work done by $\bar{R}$ in any possible displacement is zero.**
**Thus, Force of constraints does no work in any possible displacement**

# References


- T M Karade, M S Bendre : Lectures on Mechanics
  Sonu-Nilu Publication, Nagpur, Maharashtra.
- S L Loney : Statics; McMillan 7 Co., London.
- S L Loney : An Elementary Treatise on
  The Dynamics of a Particle of Rigid Bodies
  Cambridge University Press, 1956.
- H. Goldstein : Classical Mechanics
  Narosa Publishing House, New Delhi.
- B M Roy : Mechanics
  Das Ganu Prakashan, Nagpur.

# THANKS !